Efficient and fair solutions in cooperative games

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- Model: cooperative games (with transferable utilities)
- Solution: the Shapley value and more
- Axiom: efficiency & fairness-related ones
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Situation to be considered

Study solutions of cooperative games.

- Decide on how to distribute the surplus from the cooperation of multiple people to each person.
  - The surplus is limited.
  - $\rightarrow$ Someone gets more, the others get less.
  - Each wants more.
  - $\rightarrow$ Conflicts among people.
  - The surplus can only be obtained through cooperation.
  - $\rightarrow$ Coincidence of interests among players.

Our goal today

Theoretically characterize solutions of cooperative games by an efficiency and some fairness properties (axioms).

- (Pareto) efficiency:
  - The surplus obtained by people's cooperation is not left over.
- Fairness:
  - Treat people equally in some sense in distribution.
  - Various formulations have been considered.

Theorem 1 (main result)

A family of values $\leftrightarrow$ Efficiency, Indirect balanced contributions property, Symmetry for null game.

Model: cooperative games

- Players form a group (coalition) $\rightarrow$ attain the surplus.
- The surpluses of coalitions $\xrightarrow{\text{solution}}$ payoff distribution(s)

Example: a (three-person) game

Player set: $\{1, 2, 3\}$. The attainable surplus: below.

$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
\$4	\$5	\$9	\$12	\$21	\$20	\$30.

Definition: game (N, v)

- $N \subseteq \mathbb{N}$: player set (variable)
- v: $2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$: characteristic function

Solution (1/4)

- Γ: set of all games
- (One-point) solution/value φ:
  $\varphi(N, v) = (\varphi_i(N, v))_{i \in N} \in \mathbb{R}^N$ for any $(N, v) \in \Gamma$.
- family of values:
  $\{\varphi(N, v), \varphi'(N, v), \ldots\}$ for any $(N, v) \in \Gamma$.

Solution: The Shapley value (2/4)

The Shapley value (Shapley 1953) φ^{Sh}:

For any $(N, v) \in \Gamma$ and any $i \in N$,

$$\varphi_i^{Sh}(N, v) = \sum_{S \subseteq N, S \ni i} \frac{(\#S - 1)!(\#N - \#S)!}{\#N!}(v(S) - v(S \setminus \{i\})),$$

where $\#A$ is the cardinality of set A.

i's (marginal) contributions to S.

Solution: The equal divison value (3/4)

The Equal division value φ^{ED}:

For any $(N, v) \in \Gamma$ and any $i \in N$,

$$\varphi_i^{ED}(N, v) = \frac{v(N)}{\#N},$$

Solution: Numerical example (4/4)

Example: the two values φ^{Sh} and φ^{ED}

Player set: $\{1, 2, 3\}$. The attainable surplus: below.

$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
\$4	\$5	\$9	\$12	\$21	\$20	\$30.

- $\varphi^{Sh}(N, v) = (\frac{47}{6}, \frac{47}{6}, \frac{86}{6})$.
- $\varphi^{ED}(N, v) = (10, 10, 10)$.

Axioms and characterizations

- The payoff distribution determined by the solution changes as the game changes.
- i.e., Solution is a function with game as argument.
- Axiomatic characterization corresponds a solution and a collection of axioms (propoerties of solutions).

→ To answer the question, "what is an efficient and fair solution?"

Axiom: EF (1/3)

(Pareto) efficiency, EF

For any $(N, v) \in \Gamma$,

$$\sum_{i \in N} \varphi_i(N, v) = v(N).$$

- The only efficiency-related axiom in this study.
- Both φ^{Sh} and φ^{ED} satisfy EF.

Axiom: BC (2/3)

Balanced contributions, BC (Myerson 1980)

For any $(N, v) \in \Gamma$ and any $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus \{j\}, v) = \varphi_j(N, v) - \varphi_j(N \setminus \{i\}, v),$$

where in $(N \setminus \{k\}, v)$, v is restricted from 2^N to $2^{N \setminus \{k\}}$ for $k = i, j$.

j's contribution on i's payoff$=$$i$'s contribution on j's payoff

- φ^{Sh} satisfies BC but φ^{ED} does not.
- Remark 1 (Myerson 1980): $\varphi^{Sh} \leftrightarrow$ EF & BC.

Axiom: IBC (3/3)

- Is φ^{Sh} the unique efficient and fair value?

$\rightarrow$ If fair=BC, the answer is yes.

- What if fair=another requirement?
  - Especially, if fair=a weaker requirement than BC

  $\rightarrow$ more diverse discussion on efficient and fair values.

Indirect BC, IBC (Kongo 2018)

For any $(N, v) \in \Gamma$ with $\#N \geq 3$ and any $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus \{j\}, v) = \varphi_j(N, v) - \varphi_j(N \setminus \{i\}, v).$$

Combines uniform redistribution and BC.

Null player obtains zero in φ^{Sh}

- A player $k \in N$ is a null player in (N, v) if $v(S \cup \{k\}) = v(S)$ for any $S \subseteq N \setminus \{k\}$.

Reposted:The Shapley value (Shapley 1953) φ^{Sh}:

For any $(N, v) \in \Gamma$ and any $i \in N$,

$$\varphi_i^{Sh}(N, v) = \sum_{S \subseteq N, S \ni i} \frac{(\#S - 1)!(\#N - \#S)!}{\#N!}(v(S) - v(S \setminus \{i\})).$$

How can null players survive without getting anything?

Solution: The egalitarian Shapley value

Uniform redistribution may solve this problem and may improve distributional fairness.

The α-egalitarian Shapley value $\varphi^{ES,\alpha}$ (Joosten 1996):

For any $(N, v) \in \Gamma$, any $i \in N$, and any $\alpha \in \mathbb{R}$,

$$\varphi_i^{ES,\alpha}(N, v) = (1 - \alpha)\varphi_i^{Sh}(N, v) + \alpha\varphi_i^{ED}(N, v).$$

A family of values: The egalitarian Shapley values (van den Brink, Funaki, & Ju 2013): $\{\varphi_i^{ES,\alpha}(N, v) | \alpha \in [0, 1]\}$.

Solution: The egalitarian Shapley value

The α-egalitarian Shapley value $\varphi^{ES,\alpha}$ (Joosten 1996):

For any $(N, v) \in \Gamma$, any $i \in N$, and any $\alpha \in \mathbb{R}$,

$$\varphi_i^{ES,\alpha}(N,v) = (1-\alpha)\varphi_i^{Sh}(N,v) + \frac{\sum_{j\in N} \alpha\varphi_j^{Sh}(N,v)}{\#N}.$$

- Each player donates a certain percentage of one's Shapley value.
- The total donation is redistributed equally.
- BC is no longer present.

IBC reconciles uniform redistribution and BC as much as possible.

Let $x_j(N, v)$: j's donation in (N, v). Consider

$$\varphi_i^x(N, v) = \varphi_i^{Sh}(N, v) - x_i(N, v) + \frac{\sum_{j \in N} x_j(N, v)}{\#N},$$

- If $\#N = 2$, BC is obtained only when

$$x_i(N, v) = x_j(N, v) \to \varphi^x = \varphi^{Sh}.$$

- If $\#N \geq 3$, BC is obtained, i.e.,

$$x_j(N, v) = \frac{\alpha v(j)}{\#N - 1} \to x_i(N, v) \neq x_j(N, v) \to \varphi^x \neq \varphi^{Sh}.$$

$\to$ Consider BC only when $\#N \geq 3$ (=IBC).

Axioms: S^-

To clarify the solution more, add another fairness axiom.

- A game $(N, v) \in \Gamma$ is a null game if $v(S) = 0$ for any $S \subseteq N$.

Symmetry for null games, S^- (Chun 1989)

For any null game $(N, v) \in \Gamma$ and any players $\{i, j\} \subseteq N$, $\varphi_i(N, v) = \varphi_j(N, v)$.

What is the whole solution that satisfies EF, IBC & S^-?

Theorem 1 (main result): E, IBC, & S$^-$

Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. Given f and a game $(N, v) \in \Gamma$, let

$$v^f(S) = \begin{cases} f(i, v(S))v(S) & \text{if } \#N \neq 1 \text{ and } S = \{i\}, \\ v(S) & \text{otherwise,} \end{cases}$$

and let

$$\varphi^{Sh,f}(N, v) = \varphi^{Sh}(N, v^f).$$

Then, $\{\varphi^{Sh,f} |$ any f $\} \leftrightarrow$ EF, IBC & S$^-$.

- Convert games following function f.
- And apply the Shapley value to the converted game.
- A family of the Shapley value for every f is the solution.

Appendix: An essence of independence of axioms for Theorem 1

- Without EF: $\varphi_i^0(N, v) = 0$ for any $i \in N$ and any $(N, v) \in \Gamma$.
- Without IBC: $\varphi_i^{ED}(N, v) = \frac{v(N)}{\#N}$ for any $i \in N$ and any $(N, v) \in \Gamma$.
- Without S$^-$: For any game $(N, v) \in \Gamma$ that is $\#N = 1$ or $1 \notin N$, let $\hat{\varphi}(N, v) = \varphi^{Sh}(N, v)$. For any game $(N, v) \in \Gamma$ that is $\#N \geq 2$ and $1 \in N$ let

$$\hat{\varphi}_i(N, v) = \begin{cases} \varphi_i^{Sh}(N, v) - \frac{1}{\#N} & \text{if } i = 1, \text{and} \\ \varphi_i^{Sh}(N, v) + \frac{1}{\#N(\#N-1)} & \text{if } i \neq 1. \end{cases}$$

Appendix: IBC (an equivalence expression)

Indirect BC, IBC (Kongo 2018)

For any $(N, v) \in \Gamma$ and any players $\{i, j, k\} \subseteq N$,

$$\begin{aligned} &\varphi_k(N, v) - \varphi_k(N \setminus \{i\}, v) + \varphi_j(N, v) - \varphi_j(N \setminus \{k\}, v) \\ &= \varphi_k(N, v) - \varphi_k(N \setminus \{j\}, v) + \varphi_i(N, v) - \varphi_i(N \setminus \{k\}, v). \end{aligned}$$

i's contribution on k's payoff $+$ k's contribution on j's payoff
$=$ j's contribution on k's payoff $+$ k's contribution on i's payoff

Appendix: Lemma 1: EF & IBC

For any $\{i, j\} \subseteq \mathbb{N}$ and any $a \in \mathbb{R}$, let $g_i(\{i, j\}, a) \in \mathbb{R}$ and $g_j(\{i, j\}, a) \in \mathbb{R}$ satisfying

(i) $g_i(\{i, j\}, 0) + g_j(\{i, j\}, 0) = 0$, and

(ii) $g_i(\{i, j\}, a) = g_i(\{i, k\}, a) + g_k(\{j, k\}, 0)$, for any $\{i, j, k\} \subseteq \mathbb{N}$,

and let

$$\varphi_i^g(N, v) = \begin{cases} v(N) & \text{if } \#N = 1 \\ g_i(\{i, j\}, v(\{i\})) + v(\{j\}) & \\ \quad -g_j(\{i, j\}, v(\{j\})) - g_i(\{i, j\}, 0) & \text{if } \#N = 2 \\ \quad + \frac{v(\{i,j\}) - v(\{i\}) - v(\{j\})}{2} & \\ \frac{v(N) - v(N \setminus \{i\})}{\#N} + \sum_{k \in N \setminus \{i\}} \frac{\varphi_i(N \setminus \{k\}, v)}{\#N} & \text{if } \#N \geq 3. \end{cases}$$

Then, $\{\varphi^g | g \text{ satisfies (i) \& (ii).}\} \leftrightarrow$ EF & IBC.

Appendix: $\varphi^{Sh} \leftrightarrow$ EF & BC (Myerson 1980)

For the case of two-person game $(\{i,j\}, v)$,

$$\varphi_i(\{i,j\},v) + \varphi_j(\{i,j\},v) \overset{EF}{=} v(\{i,j\}), \text{and}$$

$$\begin{aligned}\varphi_i(\{i,j\},v) - \varphi_j(\{i,j\},v) &\overset{BC}{=} \varphi_i(\{i\},v) - \varphi_j(\{j\},v) \\ &\overset{EF}{=} v(\{i\}) - v(\{j\}).\end{aligned}$$

Then, $\varphi = \varphi^{Sh}$, where

$$\varphi_i^{Sh}(\{i,j\},v) = \frac{v(\{i,j\}) - v(\{i\}) - v(\{j\})}{2} + v(\{i\}),$$

$$\varphi_j^{Sh}(\{i,j\},v) = \frac{v(\{i,j\}) - v(\{i\}) - v(\{j\})}{2} + v(\{j\}).$$

Appendix: logic of uniqueness of φ

Both based on an induction w.r.t $\#$ of players in games.

Myerson (1980): EF & BC

- $\#N = 1$: EF
  $\rightarrow \varphi$ is unique.
- $\#N = k \geq 2$:
  - BC $\rightarrow k-1$ linear eqn.s
  - EF $\rightarrow 1$ linear eqn.
  - k eqn.s are independent.

$\rightarrow \varphi$ is unique.

Lemma 1: EF & IBC

- $\#N = 1$: EF
  $\rightarrow \varphi$ is unique.
- $\#N = 2$: EF (& IBC)
  $\rightarrow \varphi$ is unique w.r.t. g.
- $\#N = k \geq 3$:
  - IBC $\rightarrow k-1$ linear eqn.s
  - EF $\rightarrow 1$ linear eqn.
  - k eqn.s are independent.

$\rightarrow \varphi$ is unique w.r.t. g.

Appendix: Axioms: S

- A value in the family of Theorem 1 still allow unfairness regarding outcomes in many cases.
- To eliminate such unfairness, strengthen S^-.

- A pair $\{i, j\} \subseteq N$ are **symmetric** in (N, v) if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for any $S \subseteq N \setminus \{i, j\}$.

Symmetry, S

For any $(N, v) \in \Gamma$ and any symmetric players $\{i, j\} \subseteq N$ in it, $\varphi_i(N, v) = \varphi_j(N, v)$.

- S implies S^-.

What is the whole solution that satisfies EF, IBC & S?

Appendix: Theorem 2: E, IBC, & S.

Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. Given f and a game $(N, v) \in \Gamma$, let

$$v^f(S) = \begin{cases} f(i, v(S))v(S) & \text{if } \#N \neq 1 \text{ and } S = \{i\}, \\ v(S) & \text{otherwise,} \end{cases}$$

and let

$$\varphi^{Sh,f}(N, v) = \varphi^{Sh}(N, v^f).$$

Then, $\{\varphi^{Sh,f} |$ any f satisfying $f(i, a) = f(j, a)$ for any $i, j \in \mathbb{N}$ and any $a \in \mathbb{R} \setminus \{0\}$ $\} \leftrightarrow$ EF, IBC & S.

- The conversion of the worth of each singleton coalition is restricted to a player-independent form.

Appendix: Supplement to Theorem 2 (1/2)

Weak null player out property (NPO$^-$, van den Brink & Funaki 2009)

For any $(N, v) \in \Gamma$ any pair of players $\{i, j\} \subseteq N$, and any null player $k \in N \setminus \{i, j\}$ in (N, v),

$$\varphi_i(N, v) - \varphi_i(N \setminus \{k\}, v) = \varphi_j(N, v) - \varphi_j(N \setminus \{k\}, v).$$

Remark 2 (Kongo 2018): $\varphi^{Sh} \leftrightarrow$ EF, IBC, & NPO$^-$.

Appendix: Supplement to Theorem 2 (2/2)

NPO^- for symmetric players ($\text{NPO}^=$):

For any $(N, v) \in \Gamma$ any pair of symmetric players $\{i, j\} \subseteq N$ in (N, v), and any null player $k \in N \setminus \{i, j\}$ in (N, v),

$$\varphi_i(N, v) - \varphi_i(N \setminus \{k\}, v) = \varphi_j(N, v) - \varphi_j(N \setminus \{k\}, v).$$

Theorem 3: $\{\varphi^{Sh,f} |$ any f satisfying $f(i, a) = f(j, a)$ for any $i, j \in \mathbb{N}$ and any $a \in \mathbb{R} \setminus \{0\}$ $\} \leftrightarrow$ EF, IBC & $\text{NPO}^=$.

Appendix: Theorem 4: EF, IBC, S & H

Homogeneity, H

For any $(N, v) \in \Gamma$ and any $a \in \mathbb{R}$,
$\varphi_i(N, av) = a\varphi_i(N, v)$, where $(av)(S) = a(v(S))$ for any $S \subseteq N$.

Theorem 4: $\{\varphi^{Sh,f}$ with $f(i, a) = \alpha$ for any $i \in \mathbb{N}$ and any $a \in \mathbb{R} | \alpha \in \mathbb{R}\} \leftrightarrow$ EF, IBC, S & H.
The conversion of the worth of each singleton coalition is restricted to a player- & worth-independent form.

Appendix: Theorem 5: E, IBC, S^+, H & P

Strong symmetry (S^+, Mascler & Peleg 1966)

For any $(N, v) \in \Gamma$ and any $i, j \in N$ satisfying $v(S \cup \{i\}) - v(S) \geq v(S \cup \{j\}) - v(S)$ for any $S \subseteq N \setminus \{i, j\}$, $\varphi_i(N, v) \geq \varphi_j(N, v)$.

Positivity (P, Kalai & Samet 1987)

For any $(N, v) \in \Gamma$ satisfying $v(S) \leq v(T)$ for any $S \subseteq T \subseteq N$ and any $i \in N$, $\varphi_i(N, v) \geq 0$.

Theorem 5: $\{\varphi^{Sh,f}$ with $f(i, a) = \alpha$ for any $i \in \mathbb{N}$ and any $a \in \mathbb{R} | \alpha \in [0, 1]\} \leftrightarrow$ EF, IBC, S^+, H & P.

Appendix: Supplement to Theorem 2

BC for symmetric players (Yokote & Kongo 2017)

For any $(N, v) \in \Gamma$ and any symmetric players $\{i, j\} \subseteq N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus \{j\}, v) = \varphi_j(N, v) - \varphi_j(N \setminus \{i\}, v).$$

Theorem 2': $\{\varphi^{Sh,f} |$ any f satisfying $f(i, a) = f(j, a)$ for any $i, j \in \mathbb{N}$ and any $a \in \mathbb{R} \setminus \{0\}$ $\} \leftrightarrow$ EF, IBC & BCS.